Assignment 1.

This homework is due *Thursday* 09/30/2010. Grade for this homework will only go to *numerator* of your total grade.

1. Quick cheat-sheet.

Reminder. (Subsection 2.1.1 in textbook) On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $a+b=b+a$ for all $a,b\in\mathbb{R}$,
(A2) $(a+b)+c=a+(b+c)$ for all $a,b,c\in\mathbb{R}$,
(A3) there exists $0\in\mathbb{R}$ s.t. $0+a=a+0=a$ for all $a\in\mathbb{R}$,
(A4) for each $a\in\mathbb{R}$ there exists an element $-a$ s.t. $a+(-a)=(-a)+a=0$,
(M1) $ab=ba$ for all $a,b\in\mathbb{R}$,
(M2) $(ab)c=a(bc)$ for all $a,b,c\in\mathbb{R}$,
(M3) there exists $1\in\mathbb{R}$ s.t. $1\cdot a=a\cdot 1=a$ for all $a\in\mathbb{R}$,
(M4) for each $a\neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a\cdot(1/a)=(1/a)\cdot a=1$,
(D) $a(b+c)=ab+ac$ and $(b+c)a=ba+ca$ for all $a,b,c\in\mathbb{R}$.

Reminder. (Subsection 2.1.5 in textbook) Let $\mathbb{A}$ be a set with two operations $+$ and $\cdot$ satisfying A1–A4, M1–M3 and D. (For example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$) The set $\mathbb{P}\subset\mathbb{A}$ is called the set of *positive elements* if

(i) If $a,b\in\mathbb{P}$, then $a+b\in\mathbb{P}$,
(ii) If $a,b\in\mathbb{P}$, then $ab\in\mathbb{P}$,
(iii) If $a\in\mathbb{A}$, then exactly one of the following holds: $a\in\mathbb{P}$, $a=0$, $-a\in\mathbb{P}$.

Then $a<b$ if and only if $b-a\in\mathbb{P}$; $a\leq b$ if and only if $b-a\in\mathbb{P}\cup 0$.

One can prove the following (Theorem 2.1.7 in textbook):

Let $a,b,c\in\mathbb{A}$.

(a) If $a>b$ and $b>c$ then $a>c$,
(b) if $a>b$, then $a+c>b+c$,
(c) if $a>b$, $c>0$, then $ca>cb$,
if $a>b$, $c<0$, then $ca<cb$.

2. Exercises.

Each item (e.g. 1a, 2c, 4, 5, etc) is worth the same.

(1) (Exercises 2.1.2a, 5, 10 in textbook) For $a,b,c,d\in\mathbb{R}$, prove that
  (a) $-(a+b)=-a+(-b)$,
  (b) if $a\neq 0,\ b\neq 0$, then $1/(ab)=(1/a)(1/b)$,
  (c) if $a<b,\ c\leq d$, then $a+c<b+d$,
  (d) if $0<a<b,\ 0<c\leq d$, then $0<ac<bd$.

  Every equality and inequality you write should be accompanied by a reference to the exact property of real numbers or theorem you are using.

(2) In each case below, determine if P is a set of positive elements.
  (a) $\mathbb{A}=\mathbb{Z}$, $P=\mathbb{N}$,

(b) $\mathbb{A} = \mathbb{Z}$, $P = -\mathbb{N}$,

(c) $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > 1\}$,

(d) $\mathbb{A} = \mathbb{C}$, $P = \{z = x + iy \in \mathbb{C} : \ x > 0\}$,

(e) Prove that for $\mathbb{A} = \mathbb{C}$, there is no set of positive elements. (In other words, one cannot imbue $\mathbb{C}$ with a meaningful order.)

(3) (a) (Ex. 2.1.8a) Let x, y be rational numbers. Prove that $xy, x + y$ are rational numbers.

(b) (Ex. 2.1.8b) Let x be a rational number, y an irrational number. Prove that $x + y$ is irrational. Prove that if, additionally, $x \neq 0$, then xy is irrational.

(c) Let x, y be irrational numbers. Is it true that $x+y$ is always irrational? Is it true that $x + y$ is always rational? Same two questions for xy.

(4) (Ex. 2.2.2) If $a, b \in \mathbb{R}$, show that $|a + b| = |a| + |b|$ if and only if $ab \geq 0$.

(5) (Ex. 2.2.11) Find all $x \in \mathbb{R}$ that satisfy both $|2x - 3| < 5$ and $|x + 1| > 2$ simultaneously.

(6) (Ex. 2.2.13d) Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy $|x| - |y| \geq 2$.

(7) (a) Let $S \subset \mathbb{R}$ be a bounded set. Let $S' \subset S$ be its nonempty subset. Show that $\sup S' \leq \sup S$.

(b) (Ex. 2.3.9) Show that if A and B are bounded nonempty subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set and $\sup A \cup B = \sup\{\sup A, \sup B\}$.

(c) (Ex. 2.4.6) For A, B as in previous item, show that $A + B = \{a + b : a \in A, b \in B\}$ is a bounded set and $\sup A + B = \sup A + \sup B$.

(d) Find $\sup\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} : \ n \in \mathbb{N}\}$, $\sup\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$, $\inf\{\frac{1}{n} - \frac{1}{m} : \ m, n \in \mathbb{N}\}$.

(e) For A, B as in previous item, show that $AB = \{ab : \ a \in A, b \in B\}$ is a bounded set. Is it true that always $\sup AB = \sup A \cdot \sup B$?